2. For the function $\phi(x, y, z) = x^2 y \sin z$

   (a) Find $\nabla\phi$

   (b) Find the directional derivative $d\phi/ds$ in the direction of the (un-normalised) vector $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ at the point $(1, 2, \pi/3)$.

   (c) Find $\nabla^2\phi$.

3. [**6.6.1**] Find the gradient of $w = x^2 y^3 z$ at $(1, 2, -1)$.

6. Find $\nabla \cdot \mathbf{F}$ where $\mathbf{F} = y e^{x^2} \left(\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}} \right) + xyz\,\hat{\mathbf{k}}$

***7 In Section 3.5 of the course we studied the surface of revolution $z^2 = (x^2 + y^2)^{1/2}$. The surface element we found there can be written in Cartesian coordinates as $d\mathbf{S} = \mathbf{N}\,dx\,dy$ with

$$\mathbf{N} = \left(-\frac{x}{2}\rho^{-3/2}\,\hat{\mathbf{i}} - \frac{y}{2}\rho^{-3/2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}} \right)$$

with $\rho = \sqrt{x^2 + y^2}$.

(a) Normalise $\mathbf{N}$ to find the unit normal $\hat{\mathbf{n}}$ to this surface

(b) Consider the surface $\phi = z^2 - (x^2 + y^2)^{1/2} =$ constant. Find $\nabla\phi$ and hence calculate $\hat{\mathbf{n}} = \nabla\phi / |\nabla\phi|$. Verify that for the surface $\phi = 0$ your result agrees with that in (a).
[*If you need a vector that is normal to a given surface, it will usually be more straightforward to calculate $\nabla\phi$ than to parameterise the surface and find $\mathbf{N} = \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$. However, if you need to perform a surface integral, $\mathbf{N}$ has the advantage that its magnitude is the scale factor in $d\mathbf{S} = \mathbf{N}\,du\,dv$. On the other hand, if you already know $\hat{\mathbf{n}}$, it might be worth your while finding another way to deal with the magnitude of the parameterised area. You might also try doing this problem, including taking the gradient, directly in cylindrical polar coordinates - see handout on curvilinear coordinates.*]

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(c) Find $\nabla^2\phi$.

Q2 $\phi(x, y, z) = x^2 y \sin z$

(a) $\nabla\phi = 2xy\sin z\,\hat{i} + x^2\sin z\,\hat{j} + x^2 y\cos z\,\hat{k}$

(b) $\frac{\partial\phi}{\partial s} = \nabla\phi.\,\hat{\underline{a}} = \nabla\phi.\,\frac{\underline{a}}{|\underline{a}|}$

$\nabla\phi(1, 2, \pi/3) = 2(1)(2)\sin(\pi/3)\,\hat{i} + 1^2\sin(\pi/3)\,\hat{j} + 1^2\,2\cos(\pi/3)\,\hat{k}$

$= 4(\sqrt{3}/2)\,\hat{i} + \sqrt{3}/2\,\hat{j} + 2(1/2)\,\hat{k}$

$\underline{a} = (2, -1, 2)$ $\quad |\underline{a}| = \sqrt{1 + 2^2 + (-1)^2} = \sqrt{6}$

$\frac{\partial\phi}{\partial s} = (2\sqrt{3}, \frac{\sqrt{3}}{2}, 1) \cdot \frac{1}{\sqrt{6}}(1, 2, -1)$

$= \frac{1}{\sqrt{6}}(2\sqrt{3}, \sqrt{3}, -1) = \frac{\sqrt{6}}{6}(3\sqrt{3} - 1) = \frac{\sqrt{2}}{6}(9 - \sqrt{3})$

(c) $\nabla^2\phi = \nabla.(\nabla\phi) =$

$= \frac{\partial}{\partial x}(2xy\sin z) + \frac{\partial}{\partial y}(x^2\sin z) + \frac{\partial}{\partial z}(yx^2\cos z)$

$= 2y\sin z + 0 + x^2 y(-\sin z)$

$= (2 - x^2)\,y\sin z$

3. [**6.6.1**] Find the gradient of $w = x^2y^3z$ at $(1, 2, -1)$.

Q3. $w = x^2 y^3 z$

$$\nabla w = \frac{\partial w}{\partial x}\,\hat{i} + \frac{\partial w}{\partial y}\,\hat{j} + \frac{\partial w}{\partial z}\,\hat{k}$$

$$= 2xy^2 z\,\hat{i} + 3x^2y^2z\,\hat{j} + x^2y^3\,\hat{k}$$

@ $(1, 2, -1)$ $\quad \nabla w = 2(1)2^3(-1)\,\hat{i} + 3(1)^2 2^2(-1)\,\hat{j} + 1^2 2^3\,\hat{k}$

$$= -16\,\hat{i} - 12\,\hat{j} + 8\,\hat{k}$$

6. Find $\nabla \cdot \mathbf{F}$ where $\mathbf{F} = ye^{x^2}\left(\hat{\mathbf{i}} + 2z\,\hat{\mathbf{j}}\right) + xyz\,\hat{\mathbf{k}}$

Q6. $\nabla . F = ?$

$$F = y \exp(x^2)\,(\hat{\imath} + 2z\hat{\jmath}) + xyz\,\hat{k}$$

$$\left.\begin{aligned}
\frac{\partial F_x}{\partial x} &= 2x\,y\,\exp(x^2)\,\hat{\imath} \\
\frac{\partial F_y}{\partial y} &= 2z\,\cancel{y}\,\exp(x^2) \\
\frac{\partial F_z}{\partial z} &= xy
\end{aligned}\right\} \quad \nabla . F = (2x + 2y)\exp(x^2) + xy$$

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$$\mathbf{N} = \left(-\frac{x}{2}\rho^{-3/2}\,\hat{\mathbf{i}} - \frac{y}{2}\rho^{-3/2}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}\right)$$

with $\rho = \sqrt{x^2 + y^2}$.

(a) Normalise $\mathbf{N}$ to find the unit normal $\hat{\mathbf{n}}$ to this surface

(b) Consider the surface $\phi = z^2 - (x^2 + y^2)^{1/2} = \text{constant}$. Find $\nabla\phi$ and hence calculate $\hat{\mathbf{n}} = \nabla\phi/|\nabla\phi|$. Verify that for the surface $\phi = 0$ your result agrees with that in (a).

Q7(a) $\underline{N} = \left(-x/2\ \rho^{-3/2}\,\hat{\imath} - y/2\ \rho^{-3/2}\,\hat{\jmath} + \hat{k}\right)$

Normal to a surface $z^2 = (x^2 + y^2)^{1/2} = \rho$

$$|\underline{N}| = \sqrt{\frac{x^2}{4}\rho^{-3} + \frac{y^2}{4}\rho^{-3} + 1^2} = \sqrt{\frac{(x^2+y^2)}{4\rho^3} + 1} = \sqrt{\frac{1}{4\rho} + 1}$$

$$\Rightarrow \hat{\underline{n}} = \frac{1}{\sqrt{1 + 1/4\rho}}\left(-\frac{x}{2}\rho^{-3/2}\,\hat{\imath} - \frac{y}{2}\rho^{-3/2}\,\hat{\jmath} + \hat{k}\right)$$

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(b) $\phi = z^2 - (x^2 + y^2)^{1/2}$

$$\nabla\phi = -\frac{1}{2}(2x)(x^2+y^2)^{-1/2}\hat{i} - \frac{1}{2}(2y)(x^2+y^2)^{-1/2}\hat{j} + 2z\hat{k}$$

$$= 2z\hat{k} - \frac{(x\hat{i} + y\hat{j})}{(x^2+y^2)^{1/2}} = -\frac{x}{\rho}\hat{i} - \frac{y}{\rho}\hat{j} + 2z\hat{k}$$

$$|\nabla\phi| = \sqrt{\frac{(x^2+y^2)}{\rho^2} + 4z^2} = \sqrt{1 + 4z^2}$$

$$\Rightarrow \frac{\nabla\phi}{|\nabla\phi|} = \frac{(-x/\rho - y/\rho + 2z)}{\sqrt{1+4z^2})} \qquad \phi = 0,\ z^2 = \rho$$

$$\hat{n} = \frac{2}{\sqrt{1+4\rho}}\left(\frac{-x}{2\rho}\hat{i} - \frac{y}{2\rho}\hat{j} + \sqrt{\rho}\,\hat{k}\right)$$

$$= \underbrace{\frac{2\sqrt{\rho}}{\sqrt{1+4\rho}}}_{= \frac{2}{\sqrt{1/\rho + 4}} = \frac{1}{\sqrt{1/4\rho + 1}}}\left(-\frac{x}{2}\rho^{-3/2}\hat{i} - \frac{y}{2}\rho^{-3/2}\hat{j} + \hat{k}\right)$$